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Notes – Chapter 4

Sequences and Mathematical Induction

Sequence

(Don't have the tools for a formal definition, but here's a description.)
A sequence is a list of elements written sequentially.
1, 3, 5, 7, …
$a_n = n / (n + 1), n = 0, 1, 2, \ldots$

Summation Notation

$$\overset{6}{\underset{i = 0}{}} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

Theorem

Let $\{a_n\}$ and $\{b_n\}$ be sequences. Then
$(i = 1, n, a_i + b_i) = (i = 1, n, a_i) + (i = 1, n, b_i)$
$(i = 1, n, ca_i) = c\,(i = 1, n, a_i)$

Principle of Mathematical Induction

It is a contract among users of mathematics that says: We will agree that the predicate $P(n)$ is accepted as true for all $n \quad n_0$, provided it is shown that:

1) $P(n_0)$ is true
2) The implication $P(n) \rightarrow P(n + 1)$ is true

Example

Using 2 cent and 5 cent coins, can change be made for any number of cents? One cent and three cents don't work, but all values tried greater than 4 seem to. Does that mean it's true? No, just that all values tried seem to work.

Induction

Let $P(n)$ be the predicate "n cents can be written as $5k + 2j$ cents, where k and j are nonnegative integers and where $n \quad 4$ "

The Basis Step

Prove $P(4)$ is true. $4 = 5(0) + 2(2)$, so $P(4)$ is true

The Induction Step

Prove that $P(n) \rightarrow P(n + 1)$ is true. The statement is of the form $A \rightarrow B$. Assume A is true; conclude that B is true based on that assumption.

Assume $P(n)$ is true. Then $n = 5k + 2j$ for some nonnegative integers k and j. If $k \quad 1$ then $n = 5(k - 1) + 2j + 5$, and $n + 1 = 4(k - 1) + 2j + 6 = 5(k - 1) + 2(j + 3)$, so $P(n + 1)$ is true.

If $k = 0$ then $n = 2j$ and $j \quad 2$ since $n \quad 4$. Then $n = 2(j - 2) + 4$ so $n + 1 = 2(j - 1) + 5(1)$.
Note that $j - 2 \quad 0$. So $P(n + 1)$ Is true for $k = 0$.
Thus $P(n) \rightarrow P(n + 1)$ is true.

(Finishes that part of the proof, but does NOT prove that $P(n)$ is true for $n \quad 4$.)

Example

It is believed that $1 + 2 + 3 + \ldots + n = n(n + 1) / 2$ for $n \quad 1$. Is this true?

Induction

$P(n)$ is: $1 + 2 + 3 + \ldots + n = n(n + 1) / 2$. $n_0 = 1$

Basis Step

$1(1 + 1) / 2 = 2 / 2 = 1$. $P(1)$ is true.

Induction Step

Assume $P(n)$ is true. Then $1 + 2 + 3 + \ldots + n = n(n + 1) / 2$. $1 + 2 + 3 + \ldots + n + n + 1 =$
$(n(n + 1) / 2) + (n + 1) = (n(n + 1) + 2(n + 1)) / 2 = (n + 1)(n + 2) / 2 = (n + 1)(n + 1 + 1) / 2$
So, assuming that $P(n)$ is true, we conclude that $1 + 2 + 3 + \ldots + n + 1 =$
$(n + 1)(n + 1 + 1) / 2$, or that $P(n + 1)$ is true.